Topology and combinatorics of unavoidable complexes

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Configuration spaces in cooperative game theory and the topology of embeddings into Euclidean spaces

Abstract: There is an interesting connection between the configuration spaces arising in cooperative game theory with the complexes arising as obstructions for embedding (mapping) spaces into higher dimensional Euclidean spaces without double (multiple) points. It turns out that objects like threshold complexes and 'simple games' (von Neumann and Morgenstern) are naturally linked with Kuratowski graphs, Halin-Jung complexes, Tverberg-Van Kampen-Flores obstructions, r-unavoidable complexes, etc.

Fundamental Problem

Definition: Suppose that r and d are positive integers. A simplicial complex K is

almost r-non-embeddable

in $\mathbb{R}^d$ if and only if for each continuous map $F : K \to \mathbb{R}^d$ there exist disjoint faces $\Delta_1, \ldots, \Delta_r$ in K such that,

$$F(\Delta_1) \cap \cdots \cap F(\Delta_r) \neq \emptyset$$

Problem: Let r and d be positive integers.

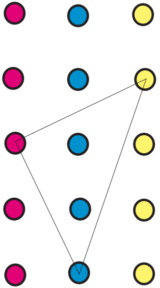
1. Characterize the simplicial complexes which are almost r-non-embeddable.
2. In particular find (interesting) sufficient conditions on K which guarantee that it is almost r-non-embeddable.

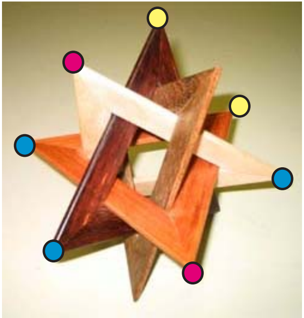
Colored Tverberg Theorem (type B)

Given 5 red, 5 blue, and 5 white points in $\mathbb{R}^3$, it is always possible to select 9 points (three in each color) and to form three triangles $\Delta_1, \Delta_2, \Delta_3$ (with vertices of different color) which have a non-empty intersection,

$$\Delta_1 \cap \Delta_2 \cap \Delta_3 \neq \emptyset .$$

(S. Vrećica, R.Ž, J. Comb. Th. A, 1993.)





Examples

$$(K_{3,3} \longrightarrow \mathbb{R}^2) \Rightarrow (2 - \text{intersection})$$

$$(K_{3,3,3} \xrightarrow{a} \mathbb{R}^2) \Rightarrow (3 - \text{intersection})$$

$$(K_{5,5,5} \longrightarrow \mathbb{R}^3) \Rightarrow (3 - \text{intersection})$$

$$(K_{4,4,4,4} \longrightarrow \mathbb{R}^3) \Rightarrow (4 - \text{intersection})$$

$$(\Delta^{n+1} \longrightarrow \mathbb{R}^n) \Rightarrow (2 - \text{intersection})$$

$$(\Delta^{(r-1)(n+1)} \longrightarrow \mathbb{R}^n) \Rightarrow (r - \text{intersection})$$

$$(\Delta^{(r-1)(n+2)}_{\lceil \frac{r-1}{r} n \rceil} \longrightarrow \mathbb{R}^n) \Rightarrow (r - \text{intersection})$$

Radon's theorem

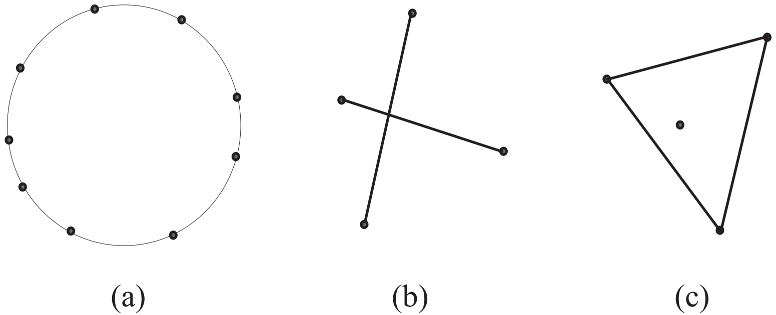


Figure: In the planar case of Radon's theorem the $(2, 2)$-partitions are persistent, while $(3, 1)$ are not.

Van Kampen-Flores theorem

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More generally, for each collection $C \subset \mathbb{R}^{2d}$ of cardinality $(2d+3)$ there exist two disjoint sub-collections C_1 and C_2 of size $\leq (d+1)$ such that,

$$\mathrm{conv}(C_1) \cap \mathrm{conv}(C_2) \neq \emptyset.$$

Van Kampen-Flores theorem
non-linear version

Theorem: For each continuous map,

$$f : \Delta_N \to \mathbb{R}^{2d}$$

where $N = 2d + 2$ and Δ_N is an N-dimensional simplex, there exist two disjoint faces σ_1 and σ_2 of Δ_N such that $\dim(\sigma_i) \leq d$ and

$$f(\sigma_1) \cap f(\sigma_2) \neq \emptyset.$$

Balanced generalized van Kampen-Flores theorem

Theorem A: Let $r \geq 2$ be a prime power, $d \geq 1$, $N \geq (r-1)(d+2)$, and $rk + s \geq (r-1)d$ for integers $k \geq 0$ and $0 \leq s < r$. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$, with $\dim \sigma_i \leq k+1$ for $1 \leq i \leq s$ and $\dim \sigma_i \leq k$ for $s < i \leq r$.

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D. Jojić, S.T. Vrećica, R.T. Živaljević.
Symmetric multiple chessboard complexes and a new theorem of Tverberg type, J. Algebraic Combin., 46 (2017), 15–31.

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[BFZ14]

1. The condition,

$$rk + s \geq (r-1)d$$

is necessary. It is equivalent to the condition that if $\sigma_1 \cap \cdots \cap \sigma_r \neq \emptyset$ in $\mathbb{R}^d$ then,

$$\dim(\sigma_1) + \cdots + \dim(\sigma_r) \geq (r-1)d.$$

2. The condition

$$N \geq (r-1)(d+2)$$

is also tight in light of Sarkaria's "type B" example (named after Karambir Sarkaria).

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(4) The case $d = 3$ of the 'sharpened van Kampen-Flores theorem' is equivalent to the Conway-Gordon-Sachs theorem which says that the complete graph K_6 on 6 vertices is 'intrinsically linked';

(5) The generalized van Kampen-Flores theorem ([BFZ14], Theorem 6.3), which improves upon earlier results of Sarkaria and Volovikov, follows for $s = 0$ and $k = \lceil \frac{r-1}{r} d \rceil$.

Decidable simple games

A simple game $\mathcal{I} \subset 2^{[n]}$ is a *decidable* simple game if

$$\text{For each} \quad A \subset [n] \quad (A \in \mathcal{I} \;\text{ or }\; A^c \in \mathcal{I})$$

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Observation: Let $\mathcal{I}$ be a simple game and let $K = 2^{[n]} \setminus \mathcal{I}$ be the associated (complementary) simplicial complex. Then $\mathcal{I}$ is both *decidable* and *non-contradictory* if and only if K is an *Alexander self dual complex*.

Weighted voting games

$$[q : w_1, w_2, \ldots, w_n]$$

$$S \subset \{1, 2, \ldots, n\}$$

is a winning (losing) coalition if and only if

$$\sum_{i \in S} w_i > q \qquad (\sum_{i \in S} w_i \leq q)$$

Simple majority game: $q = 1/2(w) = 1/2(w_1 + \cdots + w_n)$.

Definition: Threshold complex associated to the voting game $[q : w_1, w_2, \ldots, w_n]$ is the complex of all losing coalitions.

r-unavoidable complexes

Definition: Let $r \geq 2$ be an integer. Suppose that $K \subset 2^S$ is a simplicial complex with vertices in S. We say that K is r-unavoidable on S if,

$$\forall A_1, \ldots, A_r \in 2^S, \quad A_1 \uplus \ldots \uplus A_r = S \quad \Rightarrow \quad (\exists i)\, A_i \in K.$$

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P.V.M. Blagojević, F. Frick, G.M. Ziegler. Tverberg plus constraints. *B. London Math. Soc.*, 46:953–967, 2014. (‘Tverberg unavoidable complexes’)

[JVZ-3] M. Jelić, D. Jojić, M. Timotijević, S.T. Vrećica, R.T. Živaljević. Topology and combinatorics of unavoidable complexes. arXiv:1612.09487 [math.CO].

Invariant $\pi(K)$

Definition: The partition number $\pi(K)$ of a simplicial complex $K \subseteq 2^S$ is the minimum integer ν such that for each partition (disjoint decomposition) $A_1 \uplus \ldots \uplus A_\nu = S$ of S at least one of the sets A_i is in K.

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By definition the complex $K \subset 2^S$ is r-unavoidable if $\pi(K) \leq r$.

Observation: Let

$$T_{w \leq 1/r} := \{A \subset S \mid w(S) = \sum_{i \in A} w_i \leq 1/r\}$$

be the threshold complex associated with the weighted majority game $[1/r : w_1, \ldots, w_n]$ where $w_1 + \cdots + w_n = 1$. Then the complex $T_{w \leq 1/r}$ is r-unavoidable.

Collective r-unavoidable complexes

(D. Jojić, I. Nekrasov, G. Panina, R. Živaljević)

1. *Alexander r-tuples* $\mathcal{K} = \langle K_i \rangle_{i=1}^{r}$ of simplicial complexes, as a common generalization of pairs of Alexander dual complexes (Alexander 2-tuples) and r-unavoidable complexes.

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2. *Bier complexes*, defined as the deleted joins $\mathcal{K}^*_\Delta$ of Alexander r-tuples, include both standard *Bier spheres* and *optimal multiple chessboard complexes* as interesting, special cases.
3. (main results)
   - the r-fold deleted join of Alexander r-tuple is a pure complex homotopy equivalent to a wedge of spheres,
   - the r-fold deleted join of a collectively unavoidable r-tuple is $(n - r - 1)$-connected,
   - classification theorem for Alexander r-tuples and Bier complexes.

Theorem A ([JMVZ]) Let $K \subset 2^{[m]}$ be a simplicial complex. Suppose that $r = \pi(K) = p^k$ is a prime power and let $G = (\mathbb{Z}_p)^k$. Then,

$$\mathrm{Ind}_G(K^{*r}_{\Delta}) \geq m - \pi(K). \tag{1}$$

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The G-genus $\gamma_G(K)$ of K is defined as the smallest number k such that here exists a G equivariant map

$$\phi : K \to G/H_1 * \cdots * G/H_k$$

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[JMVZ] D. Jojić, W. Marzantowicz, S.T. Vrećica, R.T. Živaljević. Topology of unavoidable complexes, arXiv:1603.08472 [math.AT].

Proof methods and ideas

- $$\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}} \cong \binom{[n]}{\leqslant m_1} *_\Delta \cdots *_\Delta \binom{[n]}{\leqslant m_r},$$
  Multiple chessboard complex as the deleted join of skeletons of the simplex $\Delta([n]) \cong \Delta^{n-1}$.
- $$\Sigma(\mathcal{K}_\Delta^{*r}) := \bigcup_{\pi \in S_r} K_{\pi(1)} *_\Delta \cdots *_\Delta K_{\pi(r)} \subset [r]^{*n}$$
  Symmetrized deleted join of a collection $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ of simplicial complexes $K_i \subset 2^{[n]} = \Delta([n])$.

Theorem: ([JVZ-2]) Suppose that,

$$\Sigma = \Sigma_{n,r}^{m_1,\dots,m_s,m_{s+1},\dots,m_r;\mathbf{1}} = \Sigma_{n,r}^{\nu+1,\dots,\nu+1,\nu,\dots,\nu;\mathbf{1}}$$

is the symmetric multiple chessboard complex obtained by the S_r-symmetrization of the multiple chessboard complex $K_1 = \Delta_{n,r}^{\nu+1,\dots,\nu+1,\nu,\dots,\nu;\mathbf{1}}$ where $m_1 = \ldots = m_s = \nu + 1$ and $m_{s+1} = \ldots = m_r = \nu$. Assume that the following inequality is satisfied,

$$n \geq r(\nu + 1) + s - 1.$$

Then the complex Σ is μ-connected where,

$$\mu = m_1 + \cdots + m_r - 2 = \nu r + s - 2$$

$$\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}} \cong \binom{[n]}{\leqslant m_1} *_\Delta \cdots *_\Delta \binom{[n]}{\leqslant m_r}$$

$$\Sigma(\Delta_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}) =: \Sigma_{n,r}^{m_1,\ldots,m_r;\mathbf{1}}$$

$$\Sigma = \Sigma_{m,n}^{k_1,\ldots,k_s,k_{s+1},\ldots,k_n;\mathbf{1}} = \Sigma_{m,n}^{\nu+1,\ldots,\nu+1,\nu,\ldots,\nu;\mathbf{1}}$$

where $k_1 = \ldots = k_s = \nu + 1$ and $k_{s+1} = \ldots = k_n = \nu$, is $(\dim(\Sigma) - 1)$-connected.

Proposition: There does not exist a G-equivariant map,

$$\Sigma_{m,n}^{\nu+1,\ldots,\nu+1,\nu,\ldots,\nu;\mathbf{1}} \xrightarrow{G} (\mathbb{R}^d)^{*r}/\mathbb{R}^d$$

where $G = (\mathbb{Z}_p)^\alpha$ and $r = p^\alpha$.

Direct and indirect methods

(1) The 'direct methods' rely on a variant of *equivariant obstruction theory* and can be classified as:
   (1a) the methods which use the high connectivity of the configuration space;
   (1b) the methods involving a direct calculation of the obstruction.
(2) The 'indirect methods' comprise two basic form of reductions:
   (2a) the 'constraint method' or the Gromov-Blagojević-Frick-Ziegler reduction;
   (2b) the methods based on 'Sarkaria's index inequality' and its relatives.
(3) Ideas and tools from *linear programming, polyhedral combinatorics, and cooperative game theory.*

Constraint method

Constraint method or the Gromov-Blagojević-Frick-Ziegler reduction $\Rightarrow$ unavoidable complexes.

$$\begin{array}{ccc} K & \xrightarrow{f} & \mathbb{R}^d \\ {\scriptstyle e}\downarrow & & {\scriptstyle i}\downarrow \\ L & \xrightarrow{F} & \mathbb{R}^{d+1} \end{array}$$

Sarkaria's index inequality

$$\begin{array}{ccc} K_\Delta^{*r} & \xrightarrow{\hat{f}} & (\mathbb{R}^d)^{*r}/\mathbb{R}^d \\ \bar{e}\big\downarrow & & \big\downarrow i \\ L_\Delta^{*r} & \xrightarrow{F} & (\mathbb{R}^D)^{*r}/\mathbb{R}^D \end{array}$$

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$$\mathrm{Ind}_G(L) \geq \mathrm{Ind}_G(L_0) - \mathrm{Ind}_G(\Delta(L_0 \setminus L)) - 1$$

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(R. Ž. User's guide to equivariant methods in combinatorics, I and II. *Publ. Inst. Math. (Beograd) (N.S.)*, (I) 59(73), 1996 and (II) 64(78), 1998.)

Theorem B [JMVZ] Suppose that $r = p^k$ is a prime power and let $m_1, \ldots, m_s$ be a collection of natural numbers. Let,

$$K = K_1 * \ldots * K_s$$

be the join of a collection $\{K_i\}_{i=1}^{s}$ of simplicial complexes where each complex K_i is *r-unavoidable* on $[m_i]$. Then K is almost r-non-embeddable in $\mathbb{R}^d$ if the dimension d satisfies the inequality,

$$(r-1)(d+s+1)+1 \leq m_1 + \ldots + m_s. \qquad (3)$$

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Unavoidable vs. Non-embeddable

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Van Kampen-Flores, Grünbaum, Schild non-embedding theorem

Theorem: (G. Schild '93, B. Grünbaum '69) Let $K = K_1 * \ldots * K_s$ where each K_i is a self-dual subcomplex of the simplex $\Delta^{m_i - 1} = \Delta([m_i])$ spanned by m_i vertices. Then K is not embeddable in $\mathbb{R}^d$ where,

$$d \leq m_1 + \ldots + m_s - s - 2. \tag{4}$$

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self-dual = minimal 2-unavoidable

Minimal, non-simple, r-unavoidable!

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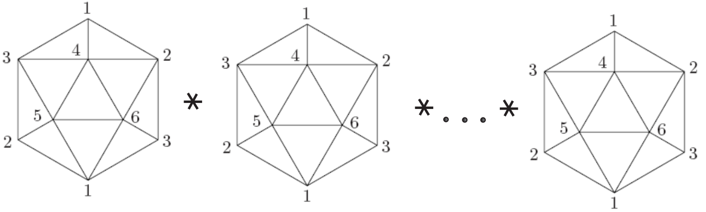


Threshold complexes

$$[q : w_1, w_2, \dots, w_n]$$

$$S \subset \{1, 2, \dots, n\}$$

is a winning (losing) coalition if and only if

$$\sum_{i \in S} w_i > q \qquad (\sum_{i \in S} w_i \leq q)$$

Definition: Threshold complex associated to the voting game $[q : w_1, w_2, \dots, w_n]$ is the complex of all losing coalitions.

Threshold complexes = weighted voting systems

$$[11 : 12, 5, 4]$$

The player with weight 12 is a 'dictator'.

$$[30 : 10, 10, 10, 9]$$

The player with weight 9 has no real power
(his/her vote is irrelevant)

$$[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

The 'voting power' distribution in UN Security Council.

r-unavoidable threshold complexes

$$\mu = (w_1, w_2, \dots, w_n)$$

$\mu \in (\mathbb{R}_+)^n$ defines a measure on $[n] = \{1, 2, \dots, n\}$

Proposition: Suppose that the total weight (mass) is,

$$\mu([n]) = w_1 + w_2 + \cdots + w_n = w.$$

Then the threshold complex,

$$T_{\mu \leq w/r} := \{S \subset [n] \mid \mu(S) \leq w/r\}$$

is r-unavoidable.

Proof: If $[n] = A_1 \uplus \cdots \uplus A_r$ then $\mu(A_i) \leq w/r$ for some i.

Theorem B revisited

Theorem B: Suppose that $r = p^k$ is a prime power and let $m_1, \ldots, m_s$ be a collection of natural numbers. Let,

$$K = K_1 * \ldots * K_s$$

be the join of a collection $\{K_i\}_{i=1}^{s}$ of simplicial complexes where each complex K_i is *r-unavoidable* on $[m_i]$. Then K is almost r-non-embeddable in $\mathbb{R}^d$ if the dimension d satisfies the inequality,

$$(r-1)(d+s+1)+1 \leq m_1 + \ldots + m_s. \qquad (5)$$

Generating r-non-embeddable complexes

$$[q; w_1, w_2, \dots, w_m]$$

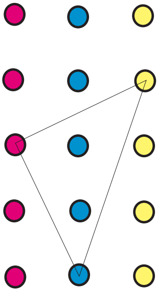
1. Choose r, quota $q = 1/r$ and the number $m = m_1$ of players;
2. Choose weights $w_1, \dots, w_{m_1}$,
3. Record the asociated r-unavoidable threshold complex K_1.
4. Repeat the procedure $s \geq 1$ times, possibly changing the number of players (and the corresponding weights);
5. The associated complexes are $K_i \subset 2^{[m_i]}$
6. Let $K = K_1 * \cdots * K_s$ be the associated join.
7. Find d from the equation,

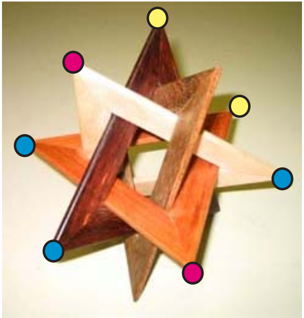
$$(r-1)(d+s+1)+1 = m_1 + \cdots + m_s$$

Then, K is almost r-non-embeddable in $\mathbb{R}^d$.

3-non-embeddablity of $K_{5,5,5}$

$[1.5; 1, 1, 1, 1, 1]$ 3 groups of players $r = 3$ $K_{5,5,5} = [5]*[5]*[5]$





(Type B Colored Tverberg Thm., S. Vrećica, R.Ž (1993))

Van Kampen-Flores Theorem

$$r = 2 \qquad q = (2n+1)/2$$

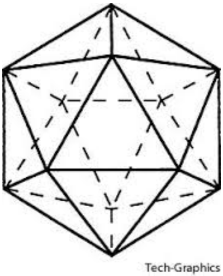
one group of (2n+1) players

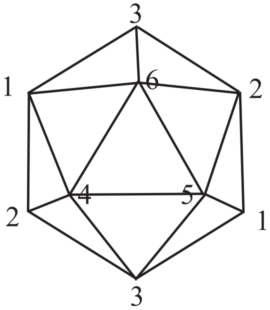
$$[(2q+1)/2; 1, 1, \ldots, 1]$$

The complex K of all losing coalitions is identified as the $(n-1)$-dimensional skeleton $\Delta_{2n}^{(n-1)}$ of the $(2n)$-dimensional simplex.

Conclusion: $\Delta_{2n}^{(n-1)}$ is not embeddable in $\mathbb{R}^{2n}$.

The 6-element triangulation $\mathbb{R}P^2_6$ of the real projective plane (hemi-icosahedron), behaves like the collection of all losing coalitions on a set of 6 players, with quota $q = 1/2$.





The conclusion is that $\mathbb{R}P^2$ is not embeddable in $\mathbb{R}^3$.

Simple vs. non-simple r-unavoidable complexes

Definition: An r-unavoidable simplicial complex $K \subset 2^{[n]}$ is *simple* if it dominates an r-unavoidable threshold complex in the sense that,

$$T_{\mu \leq 1/r} \subseteq K$$

for some probability measure $\mu \in (\mathbb{R}_+)^n$.

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Problem: Find interesting examples of non-simple r-unavoidable complexes.

Threshold characteristic $\rho(K)$

The threshold characteristic $\rho(K)$ is the maximum real number $\alpha \geq 0$ such that for some probability measure μ on $[n]$, the associated 'threshold complex' $T_{\mu<\alpha} := \{A \subset [n] \mid \mu(A) < \alpha\}$ is contained in K.

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$$\rho(K) = \sup\{\alpha \in [0, +\infty] \mid (\exists \mu \in \Delta_{n-1})\ T_{\mu \leq \alpha} \subset K\} \quad (6)$$

$$= \max\{\alpha \in [0, +\infty] \mid (\exists \mu \in \Delta_{n-1})\ T_{\mu < \alpha} \subset K\}\,. \quad (7)$$

Fundamental inequality

$$\pi(K) \leq \lfloor 1/\rho(K) \rfloor + 1$$

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(Recall that by definition if $\pi(K) = r$ then K is simple if and only if it contains a linear r-unavoidable complex $T_{\mu \leq 1/r}$.)

The problem of finding non-simple r-unavoidable complexes is reduced to finding examples such that

$$\epsilon(K) := \lfloor 1/\rho(K) \rfloor + 1 - \pi(K) > 0$$

Calculations

Proposition: If $K \subseteq 2^{[n]}$ then,

$$\rho(K) = \max_{\mu \in \Delta_{n-1}} \min_{C \notin K} \mu(C)\,. \tag{8}$$

Proposition: Let G be a group of all permutations of $[n]$ that keep the complex K invariant. Let $\Delta_{n-1}^G \subset \Delta_{n-1}$ be the closed, convex set of all G-invariant probability measures. Then,

$$\rho(K) = \max_{\mu \in \Delta_{n-1}^G} \min_{C \notin K} \mu(C)\,. \tag{9}$$

Proposition: If the action of G is transitive then,

$$\rho(K) = \min\left\{|C|/n \mid C \notin K\right\}. \tag{10}$$

Examples of non-simple r-unavoidable complexes

Proposition: The join $\mathbb{R}P^2_6 * \mathbb{R}P^2_6$ of two minimal (six vertex) triangulations of projective planes is a 3-unavoidable complex on the set $[6] \uplus [6] \cong [12]$, which is *non-simple*, i.e. it does not dominate a realizable 3-unavoidable simplicial complex $K_{\mu \leq 1/3}$.

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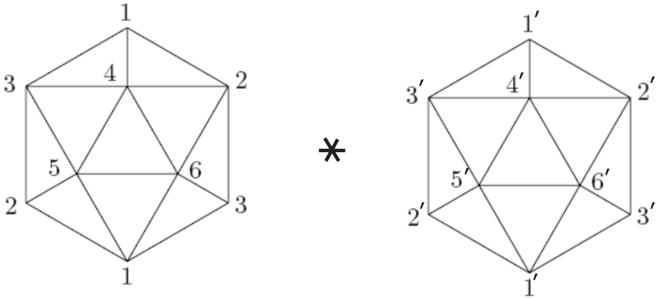

Remark: If $K \subset 2^{[n]}$ is a 2-unavoidable simplicial complex then $K * K$ is 3-unavoidable on $[n] \uplus [n] \cong [2n]$. Proposition is true for any 2-unavoidable simplicial complex K which has a 'large' group of symmetries.

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(This includes the minimal triangulation K of the complex projective plane $\mathbb{C}P^2$, etc.)

New unavoidable complexes from old

Definition:((r, s)-unavoidable complexes) Choose integers $r > s \geq 1$. A simplicial complex K is *(r, s)-unavoidable* if,

$$A_1 \uplus \ldots \uplus A_r = [m] \quad \Rightarrow \quad |K \cap \{A_i\}_{i=1}^r| \geq s.$$

In other words K is (r, s)-unavoidable if for each partition $\uplus_{i=1}^r A_i = [m]$ of $[m]$ into r non-empty sets, at least s of the sets A_i belong to K.

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r-unavoidable $\Leftrightarrow$ $(r, 1)$-unavoidable

Proposition: $K \subset 2^{[k]}$ is r-unavoidable if and only if it is $(r+k-1, k)$-unavoidable for some $k \geq 1$.

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Proof: $(\Leftarrow)$
If $A_1 \uplus \ldots \uplus A_r = [n]$ let $A_1 \uplus \cdots \uplus A_r \uplus \emptyset \cdots \uplus \emptyset = [n]$ be a new partition of size $r+k-1$.

$(\Rightarrow)$
If $A_1 \uplus \ldots \uplus A_{r+k-1} = [n]$ let $B_1 \uplus \cdots \uplus B_r = [n]$ be a new partition of size r where $B_i = A_i$ for $1 \leq i \leq r-1$ and $B_r = \cup_{j \geq r} A_j$, etc. $\square$

Proposition: Suppose that $K \subset 2^S$ and $L \subset 2^T$ are simplicial complexes on disjoint sets S and T. Assume that K is s-unavoidable on S and L is (r, s)-unavoidable on T. Then the join $K * L$, interpreted as a subcomplex of $2^{S \cup T}$, is r-unavoidable.

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Proof: Let $C_1 \uplus \cdots \uplus C_r = S \cup T$ where $C_i = A_i \cup B_i$, $A_1 \uplus \cdots \uplus A_r = S$ and $B_1 \uplus \cdots \uplus B_r = T$.

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Since L is (r,s)-unavoidable on T there exist distinct indices $\{i_k\}_{k=1}^{s}$ such that $B_{i_k} \in L$ for each k. Since K is s-unavoidable there exists k such that $A_{i_k} \in K$ and as a consequence $C_{i_k} = A_{i_k} \cup B_{i_k} \in K * L$. $\square$

Corollary: Suppose that $K \subset 2^S$ and $L \subset 2^T$ are simplicial complexes on disjoint sets S and T. Assume that K is 2-unavoidable on S and L is $(r-1)$-unavoidable on T. Then the join $K * L$ is r-unavoidable on $S \cup T$.

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Corollary: If K is a self-dual (minimal 2-unavoidable) subcomplex of 2^S then $K * \cdots * K = K^{*r-1}$ is an r-unavoidable complex on $S \times [r]$.

Minimal, non-simple, r-unavoidable!

Theorem: Suppose that X_i (for $i = 1, \ldots, r-1$) is either the 6-element triangulation of $\mathbb{R}P^2$ or the 9-element triangulation of $\mathbb{C}P^2$. Then the complex,

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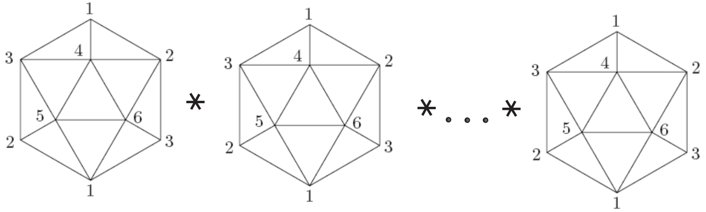

Other properties of $L = \mathbb{R}P^2 * \mathbb{R}P^2$

Let,

$$L^{*3}_{\Delta} \cong L^{*3}_{\Delta} \cong (\mathbb{R}P^2)^{*3}_{\Delta} * (\mathbb{R}P^2)^{*3}_{\Delta}$$

be the 3-fold deleted join of L. Then,

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2. $$\mathrm{Ind}_{\mathbb{Z}_3}(L^{*3}_{\Delta}) \geq 9$$
3. The complex L^{*3}_{Δ} is 8-connected.
4. There does not exist a $\mathbb{Z}_3$-equivariant map, $f : L^{*3}_{\Delta} \to X$, for each free, $\mathbb{Z}_3$-complex X of dimension ≤ 8.

Key examples from [BFZ]

Example: (The key example (iv) from [BFZ])
If $r(k+1)+s > N+1$ with $0 \leq s \leq r$, then the complex

$$K = \Delta_N^{(k-1)} \cup \Delta_{N-(r-s)}^{(k)} = \binom{[N+1]}{\leq k} \cup \binom{[N+1-(r-s)]}{\leq k+1} \subset 2^{[N+1]}$$

is r-unavoidable on $[N+1]$.

Proposition [JJTVZ] The complex $K = \Delta_N^{(k-1)} \cup \Delta_{N-(r-s)}^{(k)}$ is intrinsically linear, r-unavoidable complex.

Proof:

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Proposition [JJTVZ] The complex $K = \Delta_N^{(k-1)} \cup \Delta_{N-(r-s)}^{(k)}$ is intrinsically linear, r-unavoidable complex.

Proof: By solving a linear program evaluating (estimating) $\rho(K)$ for the complex $K = \binom{[n]}{\leq k} \cup \binom{[p]}{\leq k+1}$.

K	$[3]$	$\mathbb{R}P^2_6$	$\mathbb{C}P^2_9$	$\mathbb{H}P^2_{15}$	$\mathfrak{R}_3$
$\pi(K^{*n})$	$n+1$	$n+1$	$n+1$	$n+1$	$n+1$
$\rho(K^{*n})$	$2/3n$	$1/2n$	$4/9n$	$6/15n$	$\leq 6/15n$
$\epsilon(K^{*n})$	$\lfloor n/2 \rfloor$	n	$\lfloor 5n/4 \rfloor$	$\lfloor 3n/2 \rfloor$	?
$\nu(K^{*n})$	$3n$	$6n$	$9n$	$15n$	$15n$
ϵ/ν	$\approx 1/6$	$1/6$	$\approx 5/36$	$\approx 1/10$	?

(11)

The complex $\mathfrak{R}_3$

Example: Let $\mathfrak{R}_3$ be the the complex of all graphs $\Gamma \subset K_6$ such that the complement $\Gamma^c = V \setminus \Gamma$ contains a triangle K_3. Then,

$$\pi(\mathfrak{R}_3) = 2 \qquad \rho(\mathfrak{R}_3) = 6/15 \qquad \nu(\mathfrak{R}_3) = 15. \tag{12}$$

The complex $\mathfrak{K}_3$

Example: Let $\mathfrak{K}_3$ be the the complex of all graphs $\Gamma \subset K_6$ such that the complement $\Gamma^c = V \setminus \Gamma$ contains a triangle K_3. Then,

$$\pi(\mathfrak{K}_3) = 2 \qquad \rho(\mathfrak{K}_3) = 6/15 \qquad \nu(\mathfrak{K}_3) = 15. \tag{12}$$

Proof:

$$\rho(\mathfrak{K}_3) = \min\{|A|/15 \mid A \notin \mathfrak{K}_3\}. \tag{13}$$