Chessboard complexes

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- The early examples include the fundamental theorem of algebra, Brouwer's fixed point theorem and the domain invariance theorem; the ham-sandwich theorem.
- There are now many applications in other areas of Mathematics and sciences in general (the shape recognition, topological robotics, motion planning algorithms, topological complexity, topological data analysis, topological analysis of neural networks).

On applications of Algebraic topology

- Some of the applications were quite unexpected, such as Lovász proof of Kneser conjecture, saying that the chromatic number of Kneser graph (with vertices $\binom{[2n+k]}{n}$) $K_{n,k}$ equals $k+2$.

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- **Theorem.** (D. Gale) If C is a compact convex set in $\mathbb{R}^d$ of the width w and C' is the image of the injective, continuous Lipschitz mapping (with Lipschitz constant L) $f : C \to \mathbb{R}^d$, then the width of the set C' is at most Lw.

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- Both results depend on topological methods, in particular Borsuk-Ulam theorem.

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- It has several equivalent formulations and relatives and many consequences.
- There is no antipodal ($\mathbb{Z}/2$-equivariant) mapping $S^{n+1} \to S^n$, or $X \to Y$ if Y is n-dimensional and X is n-connected and the action is free.

On applications of Algebraic topology

- **Theorem.** (R. Živaljević, S. V., 1990) For every collection $\mu_1, ..., \mu_k$ of probability measures on $\mathbb{R}^d$, there is a $(k-1)$-dimensional flat F so that every half-space H_+ containing F satisfies $\mu_i(H_+) \geq \frac{1}{d-k+2}$ for every $i = 1, ..., k$.

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- The special case $k = 1$ reduces to the well-known Rado's center point theorem, and the special case $k = d$ reduces to the ham-sandwich theorem.

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- For the polynomials in the variables $x_1, \ldots, x_n, y_1, \ldots, y_k$ the ideal generated by the symmetric polynomials in all $n + k$ variables does not contain the monomial $(x_1 x_2 \cdots x_n)^k$.

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- The first examples: $\Delta_{3,2}$ is a hexagon, $\Delta_{4,3}$ is a torus.

The first examples

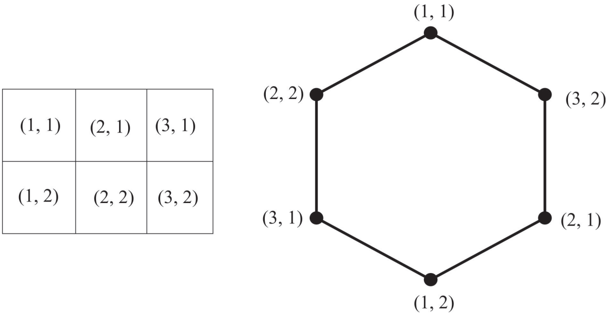

Figure: $\Delta_{3,2}$ is a hexagon

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(1, 1)	(2, 1)	(3, 1)	(4, 1)
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(1, 3)	(2, 3)	(3, 3)	(4, 3)

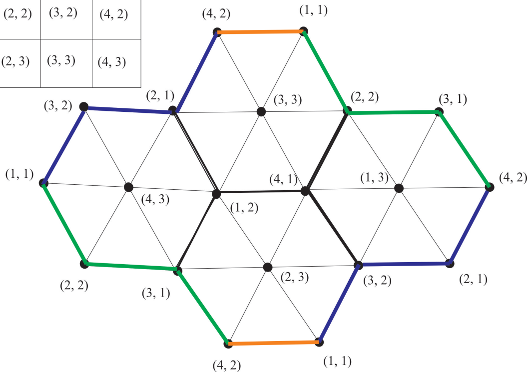

Figure: $\Delta_{4,3}$ is a torus

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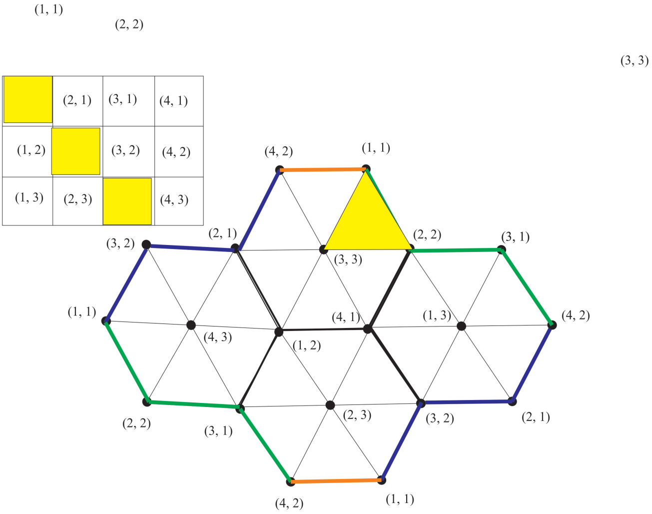

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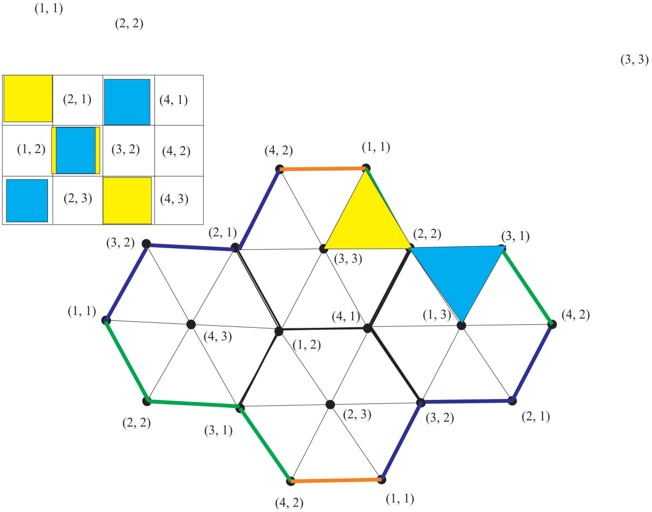

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coset complex of certain subgroups in the symmetric group

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n*-fold* 2*-deleted join of vertices of the* $(m-1)$*-simplex*

$$\Delta_{m,n} = [m]^{*n}_{\Delta(2)} = \left((\sigma^{m-1})^{(0)}\right)^{*n}_{\Delta(2)} = \left([1]^{*m}_{\Delta(2)}\right)^{*n}_{\Delta(2)}$$

Properties of chessboard complexes are important!

- **Colored Tverberg theorem.** (R. Živaljević, S. V., 1992) For r prime and any $d+1$ collections (colors) of finite sets of $2r-1$ points each in $\mathbb{R}^d$, there are r disjoint sets each containing at most one point of every color so that their convex hulls intersect.

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- If such r sets do not exist, we have the mapping from the configuration space of joins of r-tuples of disjoint simplices to the join of r copies of $\mathbb{R}^d$ missing the diagonal.
- Simplices with one vertex of each color $[2r-1]^{*(d+1)}$.
- Collections of r vertex-disjoint simplices with at most one vertex of each color could be described as $\left([2r-1]^{*(d+1)}\right)^{*r}_{\Delta} = ([2r-1]^{*r}_{\Delta})^{*(d+1)} = (\Delta_{r,2r-1})^{*(d+1)}$.

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- $\left(\mathbb{R}^d\right)^{*r}_{\Delta} \simeq \mathbb{R}^{rd+r-1} \setminus \mathbb{R}^d \simeq \mathbb{R}^{rd+r-d-1} \setminus \{0\} \simeq S^{(r-1)(d+1)-1}$.

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- For a prime r **there is no** $\mathbb{Z}_r$-map $(\Delta_{r,2r-1})^{*(d+1)} \to \left(\mathbb{R}^d\right)^{*r}_{\Delta}$.

Properties of chessboard complexes are important!

- The colored Tverberg theorem was used to establish the halving plane theorem, the point selection theorem, the hitting set theorem, the weak ϵ-net theorem. (N. Alon, I. Bárány, Z. Füredi, D. Kleitman, L. Lovász)

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- This result is improved by P. Blagojević, B. Matschke, G. Ziegler in 2009. by proving that r points of each color is sufficient when $r+1$ is prime, establishing in this way the original conjecture by I. Bárány and D. Larman in this special case.

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- **Theorem.** (S. V., R. Živaljević, 2009) The degree of each $\mathbb{Z}/r$-equivariant map $f : (\Delta_{r,r-1})^{*d} \to S(W_r^{\oplus d})$ satisfies $\deg(f) \equiv_{\mathrm{mod}\, r} (-1)^d$, provided r is a prime number.

Some generalizations

- A. Björner, L. Lovász, S. V., R. Živaljević proved in 1994 the general lower bound on the connectivity of chessboard complexes and also for some of their generalizations (obtained from higher-dimensional chessboards, matching complexes of complete multipartite hypergraphs etc.). Some other properties and invariants of these complexes were considered.

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- It was later proved that these estimates are sharp. (J. Shareshian, M. Wachs)

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- For every constellation of five red, five blue and five white stars in the space, there exist three vertex disjoint triangles formed by stars of different colors which have a nonempty intersection.

Symmetric homology of algebras

- Considering the symmetric analogue of the cyclic homology of algebras, S. Ault, Z. Fiedorowicz wanted to show that there was a spectral sequence converging strongly to $HS_*(A)$ with the E^1-term

$$E^1_{p,q} = \bigoplus_{\overline{u} \in X^{p+1}/S_{p+1}} \widetilde{H}_{p+q}(EG_{\overline{u}} \ltimes_{G_{\overline{u}}} N\mathcal{S}_p / N\mathcal{S}'_p \,;\, k).$$

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- The fact that the connectivity of the space $N\mathcal{S}_p / N\mathcal{S}'_p$ is an increasing function of p is crucial to show this convergence.

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- The subcomplex Ω_{p+1} is obtained from the chessboard complex $\Delta_{p+1,p+1}$ by deleting the simplices of the form $((x_{\sigma(1)}, x_{\sigma(2)}), (x_{\sigma(2)}, x_{\sigma(3)}), ..., (x_{\sigma(k)}, x_{\sigma(1)}))$ for any $k \in \{1, ..., p+1\}$ and any permutation σ (cycles).

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- We proved that $Sym_*^{(p)}$ was $\left[\frac{2}{3}(p-1)\right]$-connected.

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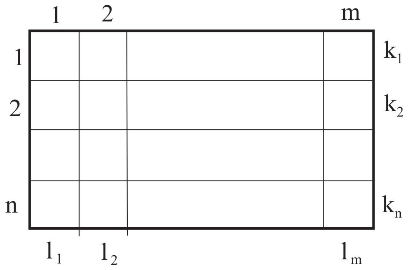

Figure: $\Delta_{m,n}^{k_1,\dots,k_m;l_1,\dots,l_m}$

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- The first examples:

$$\Delta^{2;1}_{3,2} \approx S^1 \times D^1, \ \Delta^{2,1;1}_{4,2} \approx S^2, \ \Delta^{2;1}_{5,2} \approx S^3.$$

The new examples

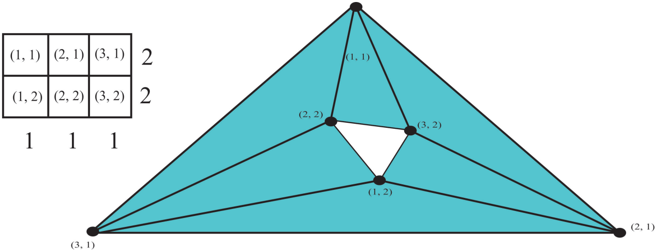

Figure: $\Delta^{2,1}_{3,2}$ is a triangulation of cylinder

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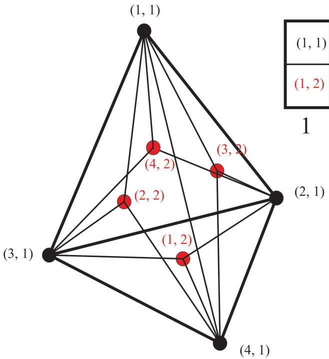

(1, 1)	(2, 1)	(3, 1)	(4, 1)	2
(1, 2)	(2, 2)	(3, 2)	(4, 2)	1
1	1	1	1	

Figure: $\Delta^{2,1;1}_{4,2} \cong \mathbb{S}^2$

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- link of a 2-simplex in $\Delta_{5,2}^{2,1}$ is a set of two points
- $\Delta_{5,2}^{2,1}$ is a 2-connected, simplicial 3-manifold.

The appearances

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- Establishing the topological properties of these complexes was our main motivation.

Topological properties

- **Theorem.** (D. Jojić, S. V., R. Živaljević, 2018) If $k_1 + \cdots + k_n \leq l_1 + \cdots + l_m - n + 1$, the multiple chessboard complex $\Delta_{m,n}^{k_1,\dots,k_n;l_1,\dots,l_m}$ is $(k_1 + \cdots + k_n - 2)$-connected.

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- **Corollary.** By replacing rows and columns, we see that if $l_1 + \cdots + l_m \leq k_1 + \cdots + k_n - m + 1$, the same complex is $(l_1 + \cdots + l_m - 2)$-connected.
  If $k_1 = \cdots = k_n = k$ and $l_1 = \cdots = l_m = l$, we obtain the chessboard complex $\Delta_{m,n}^{k;l}$, and it follows that this complex is $(kn - 2)$-connected if $kn \leq lm - n + 1$, and it is $(lm - 2)$-connected if $lm \leq kn - m + 1$.

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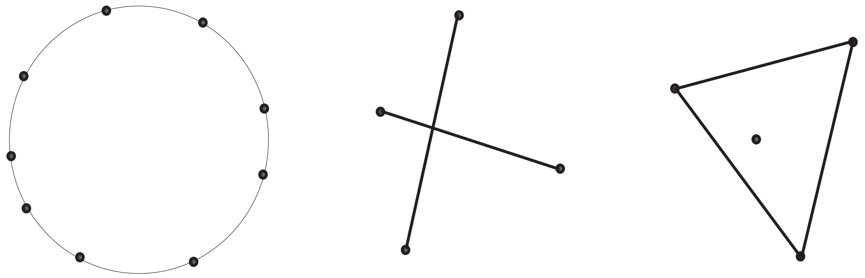
- Van Kampen-Flores theorem is an example of the result of this type, saying that for each continuous map $f : \Delta_N \to \mathbb{R}^{2d}$, where $N = 2d + 2$ and Δ_N is an N-dimensional simplex, there exist two disjoint faces σ_1 and σ_2 of Δ_N such that $\dim(\sigma_i) \leq d$ and $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.

Radon's theorem



(a) (b) (c)

Figure: In the planar case of Radon's theorem the $(1, 1)$-partitions are persistent, while $(2, 0)$ are not.

The applications - Van Kampen

- P. Blagojević, F. Frick and G. Ziegler raised a conjecture that, under some hypothesis, there are r disjoint faces of a simplex whose dimensions are two consecutive integers and whose images intersect.

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- P. Blagojević, F. Frick and G. Ziegler raised a conjecture that, under some hypothesis, there are r disjoint faces of a simplex whose dimensions are two consecutive integers and whose images intersect.
- **Theorem.** (D. Jojić, S. V., R. Živaljević, 2017) Let $r \geq 2$ be a prime power, $d \geq 1$, $N \geq (r-1)(d+2)$, and $rk + s \geq (r-1)d$ for integers $k \geq 0$ and $0 \leq s < r$. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$, with $\dim\, \sigma_i \leq k+1$ for $1 \leq i \leq s$ and $\dim\, \sigma_i \leq k$ for $s < i \leq r$.

Symmetrized multiple chessboard complex

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- In order to have a group action of permuting the rows, we have to deal with a symmetrized multiple chessboard complexes

$$\Sigma_{N+1,r}^{k_1,\dots,k_r;1} = S_r \cdot \Delta_{N+1,r}^{k_1,\dots,k_r;1} = \bigcup_{\sigma \in S_r} \Delta_{N+1,r}^{k_{\sigma(1)},\dots,k_{\sigma(r)};1},$$

where $k_1, \dots, k_r = k+2, \dots, k+2, k+1, \dots, k+1$.

Symmetrized multiple chessboard complex

- If $r = p^\alpha$ is a prime power, there is a fixed point free action of the group $(\mathbb{Z}/p)^\alpha$ on the complex $\Sigma_{m,r}^{k_1,\dots,k_r;1}$.

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- If such r faces does not exist, we obtain equivariant mapping of this complex to the representation sphere of appropriate dimension.

Consequences

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(5) The generalized van Kampen-Flores theorem which improves upon earlier results of Sarkaria and Volovikov, follows for $s = 0$ and $k = \lceil \frac{r-1}{r} d \rceil$.

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- **Theorem.** Let r be a prime power, and let $(k+1)r + r - 1 \leq (N+1)(j-1)$ and $(r-1)(d+1)+1 \leq r(k+1)$. Then for every continuous mapping from Δ^N to $\mathbb{R}^d$ there are r j-wise disjoint faces of the simplex Δ^N of dimension at most k whose images have nonempty intersection.

The applications - colored Tverberg

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- We could prove a colored Tverberg type theorem, allowing more than 1 vertex of the same color in each face, and we consider j-wise disjoint faces.
- **Theorem.** (D. Jojić, S. V., R. Živaljević, 2018) Let r be a prime power. Given k finite sets of points in $\mathbb{R}^d$ (called colors), of m points each, so that $pr \leq m(j-1) - r + 1$ and $(r-1)(d+1) + 1 \leq prk$, it is possible to divide the points in r j-wise disjoint sets containing at most p points of each color, so that their convex hulls intersect.

The applications - colored Tverberg

- **Theorem.** (D. Jojić, S. V., R. Živaljević, 2018) Let r be a prime power. Given k finite sets of points in $\mathbb{R}^d$ (called colors), of m points each, so that $jm - 1 \leq pr$ and $(r-1)(d+1)+1 \leq (j-1)mk$, it is possible to divide the points in r j-wise disjoint sets containing at most p points of each color, so that their convex hulls intersect.

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- Let us consider the very special case $p = 1$ and $j = 2$.

The applications - colored Tverberg

- **Theorem.** Let r be a prime power. Given k finite sets of points in $\mathbb{R}^d$ (called colors), of m points each, so that $2m - 1 \leq r$ and $(r-1)(d+1) + 1 \leq mk$, it is possible to divide the points in r pairwise disjoint sets containing at most 1 point of each color, so that their convex hulls intersect.

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- **Theorem.** Let r be a prime power. Given k finite sets of points in $\mathbb{R}^d$ (called colors), of m points each, so that $2m - 1 \leq r$ and $(r-1)(d+1)+1 \leq mk$, it is possible to divide the points in r pairwise disjoint sets containing at most 1 point of each color, so that their convex hulls intersect.
- It is easy to see that the assumptions on the total number of points is the best possible, since the set of $(r-1)(d+1)$ points in the general position could not be divided in r disjoint sets whose convex hulls intersect.

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THANK YOU FOR YOUR ATTENTION!